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On the relation between Leon's extension and the $\bar{\partial}$-dressing with variable normalization

Wang Ning and Li Qi

Department of Physics, Ocean University of Qingdao, 5-Yushan Road, Qingdao 266003, People's Republic of China

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Abstract

It is proved that for nonlinear evolution with a singular dispersion relation, Leon's extension of the inverse spectral transform is equivalent to the $\bar{\partial}$-dressing problem with variable normalization, at least in a subclass of dispersion relations.

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1. Introduction

The $\bar{\partial}$-dressing method proposed by Zakharov and Shabat has been used successfully to solve (1 + 1)- and (2 + 1)-dimensional nonlinear evolutions [1]. Several extensions of the $\bar{\partial}$-dressing method have been considered [2–4]. Among these, the spectral transform method which includes nonlinear evolution when the (x, t) dependence of the spectral transform is arbitrary, was developed by Leon [5]. The method was shown to be effective, and was used to derive a new class of solutions to the nonlinear Schrödinger equation, and to solve initial boundary value problems for resonant wave coupling processes in stimulated Brillouin scattering of plasma waves and stimulated Raman scattering in nonlinear optics [6–9].

The main purpose of this work is to answer the following simple problem: are there some relations between Leon's extension and the $\bar{\partial}$-dressing method with variable normalization? We shall prove, in particular, that Leon's result can be easily extended to a 2 + 1 non-local $\bar{\partial}$-problem, and for nonlinear evolution having a singular dispersion relation Leon's extension of the inverse spectral transform is equivalent to a $\bar{\partial}$-dressing problem with variable normalization.

At first we briefly recall Leon's idea. For the (1 + 1)-dimensional local $\bar{\partial}$-problem

$$\frac{\partial\chi}{\partial\bar{\lambda}}(x,\lambda) = \chi(x,\lambda)\,R(x,\lambda). \tag{1}$$

Leon's extension from the usual $\bar{\partial}$-dressing approach is that the homogeneous time–spatial evolutions for spectral transform operator R

$$\begin{aligned}\frac{\partial}{\partial x}R &= [R,\Lambda]\\ \frac{\partial}{\partial t}R &= [R,\Omega]\end{aligned} \tag{2}$$

are replaced by the inhomogeneous time–spatial relations

$$\begin{aligned}\frac{\partial}{\partial x}R &= [R, \Lambda] + N \\ \frac{\partial}{\partial t}R &= [R, \Omega] + M\end{aligned} \tag{3}$$

where Λ and Ω are matrix-valued dispersion operators which determine the dispersion relation of the corresponding spectral problem. It can be proved that the matrices N and M obey the compatibility constraint

$$N_t - [N, \Omega] - M_x + [M, \Lambda] = [\Lambda_t - \Omega_x + [\Omega, \Lambda], R]. \tag{4}$$

It was shown by Leon that equation (4) is satisfied if N and M satisfy

$$N = \frac{\partial}{\partial \bar{\lambda}}\Lambda \qquad M = \frac{\partial}{\partial \bar{\lambda}}\Omega \tag{5a}$$

and

$$\Lambda_t - \Omega_x + [\Omega, \Lambda] = 0. \tag{5b}$$

In particular, for the case where the dispersion operators Λ and Ω take polynomial form, the matrices N and M vanish.

2. Leon's extension of the 2 + 1 $\bar{\partial}$-dressing problem

Instead of equations (1) and (3) for the (1 + 1)-dimensional problem, we start from the general 2 + 1 non-local $\bar{\partial}$-problem with variable normalization (the variable x denotes the 3-vector x_j, $j = 1, 2, 3$)

$$\frac{\partial \chi}{\partial \bar{\lambda}}(\lambda, \bar{\lambda}; x) = \frac{\partial \eta}{\partial \bar{\lambda}} + \hat{R} * \chi \tag{6}$$

where the spectral transform operator $\hat{R}$ is a linear integral operator acting like

$$(\hat{R} * \chi)(\lambda, \bar{\lambda}) = \iint_c \mathrm{d}\mu \wedge \mathrm{d}\bar{\mu}\, \chi(\mu, \bar{\mu})\, R(\mu, \bar{\mu}; \lambda, \bar{\lambda}; x) \tag{7}$$

and $\eta(\lambda)$ is, in general, an arbitrary rational function of λ. The role of this function is to fix the normalization of the function χ and then to make the problem's solution uniquely determined.

By analogy with the 1 + 1 problem, the time and space dependence of the 2 + 1 spectral transform is defined by

$$\frac{\partial}{\partial x_i}\hat{R} = [\hat{R}, \hat{B}_i^*] + \hat{N}_i \qquad i = 1, 2, 3 \tag{8}$$

where $\hat{B}_i^*$, $i = 1, 2, 3$, are the adjoint operators of $\hat{B}_i$ with respect to the standard bilinear form $\langle f, g\rangle = \int \mathrm{d}\xi\ \mathrm{tr}(f(\xi)g(\xi))$. These operators act as

$$(\hat{B}_i \chi)(\lambda, \bar{\lambda}) = \iint_c \mathrm{d}\mu \wedge \mathrm{d}\bar{\mu}\, B_i(\lambda, \bar{\lambda}; \mu, \bar{\mu}; x)\, \chi(\mu, \bar{\mu}) \qquad i = 1, 2, 3 \tag{9a}$$

$$(\hat{B}_i^* \chi)(\lambda, \bar{\lambda}) = \iint_c \mathrm{d}\mu \wedge \mathrm{d}\bar{\mu}\, \chi(\mu, \bar{\mu})\, B_i(\mu, \bar{\mu}; \lambda, \bar{\lambda}; x) \tag{9b}$$

respectively.

From (8), it is clear that the operators $\hat{N}_i$, $i = 1, 2, 3$, satisfy the following compatibility constraints:

$$\frac{\partial}{\partial x_j}\hat{N}_i - [\hat{N}_i, \hat{B}_j^*] - \frac{\partial}{\partial x_i}\hat{N}_j + \left[\hat{N}_j, \hat{B}_i^*\right] + \left[\hat{R}, \frac{\partial}{\partial x_j}\hat{B}_i^* - \frac{\partial}{\partial x_i}\hat{B}_j^* - \left[\hat{B}_i^*, \hat{B}_j^*\right]\right] = 0. \qquad (10)$$

As equation (10) with (8) is nonlinear for $\hat{N}_i$, $i = 1, 2, 3$, it is difficult to find the general solution, but the following special solutions can be easily found.

Proposition 1. *If the dispersion operators satisfy*

$$\frac{\partial}{\partial x_j}\hat{B}_i^* - \frac{\partial}{\partial x_i}\hat{B}_j^* + [\hat{B}_j^*, \hat{B}_i^*] = 0 \qquad i \neq j = 1, 2, 3 \qquad (11)$$

then equation (10) has the solutions

(a) $$\hat{N}_i = \frac{\partial}{\partial \bar{\lambda}}\hat{B}_i^* \qquad i = 1, 2, 3. \qquad (12)$$

(b) $$\hat{N}_i = \mathrm{e}^{-F(x)}\hat{N}_i^0\mathrm{e}^{F(x)} \qquad \frac{\partial}{\partial x_j}\hat{N}_i^0 = 0 \qquad i, j = 1, 2, 3 \qquad (13)$$

satisfying equations

$$\frac{\partial}{\partial x_j}\hat{N}_i = [\hat{N}_i, \hat{B}_j^*] \qquad i, j = 1, 2, 3 \qquad (14)$$

(c) $$\hat{N}_i = \mathrm{e}^{-F(x)}\frac{\partial}{\partial x_i}M\mathrm{e}^{F(x)} \qquad (15)$$

in which M is an arbitrary matrix function of the variables x_j $(j = 1, 2, 3)$ and of the spectral parameter λ. The operator $F(x) \equiv \sum_{i=1}\partial_i^{-1}\hat{B}_i^*$ is a matrix-valued operator of the physical variables x_i $(i = 1, 2, 3)$ and of the spectral parameter λ.

Proof.

Case (a). A simple calculation shows that

$$\frac{\partial}{\partial x_j}\hat{N}_i - [\hat{N}_i, \hat{B}_j^*] - \frac{\partial}{\partial x_i}\hat{N}_j + [\hat{N}_j, \hat{B}_i^*] + \left[\hat{R}, \frac{\partial}{\partial x_j}\hat{B}_i^* - [\hat{B}_i^*, \hat{B}_j^*] - \frac{\partial}{\partial x_i}\hat{B}_j^*\right]$$
$$= c\frac{\partial}{\partial\bar{\lambda}}\left[\frac{\partial}{\partial x_j}\hat{B}_i^* - [\hat{B}_i^*, \hat{B}_j^*] - \frac{\partial}{\partial x_i}\hat{B}_j^*\right]$$
$$+\left[\hat{R}, \frac{\partial}{\partial x_j}\hat{B}_i^* - \frac{\partial}{\partial x_i}\hat{B}_j^* - [\hat{B}_i^*, \hat{B}_j^*]\right] = 0. \qquad (16)$$

Case (b). It is easily verified that equation (10) is satisfied by virtue of (14) and, that the general solution of (14) is given by (13).

Case (c). When (11) is satisfied, equation (10) becomes

$$\frac{\partial}{\partial x_j}\hat{N}_i - [\hat{N}_i, \hat{B}_j^*] - \frac{\partial}{\partial x_i}\hat{N}_j + [\hat{N}_j, \hat{B}_i^*] = 0. \tag{17}$$

Inserting the ansatz

$$\hat{N}_i = e^{-F(x)}\, \hat{N}_i^0(x_i, \mu, \lambda)\, e^{F(x)}$$

into (17) yields

$$\frac{\partial}{\partial x_j}\hat{N}_i^0 = \frac{\partial}{\partial x_i}\hat{N}_j^0 \qquad \text{that is} \quad \hat{N}_i^0 = \frac{\partial}{\partial x_i}M. \tag{18}$$

□

Equation (11) itself is a nonlinear equation which has to be solved to find the dispersion relation first. When the operators $\hat{B}_j$ reduce to multiplication by matrices, the following relations:

$$\frac{\partial}{\partial x_i}\hat{B}_j^* = 0 \qquad [\hat{B}_i^*, \hat{B}_j^*] = 0 \qquad i \neq j, 1, 2, 3 \tag{19}$$

provide a solution to (11). Then

$$\hat{B}_j^* = K(x, \lambda)^{-1}\frac{\partial}{\partial x_j}K(x, \lambda) \tag{20}$$

where $K(x, \lambda)$ is an arbitrary matrix-valued function of the physical variables and the spectral parameter. In particular, equation (19) can be considered as a special case of the general case (20) when we select

$$K = \exp\left(\sum_{i=1} \hat{B}_i^* x_i\right) \qquad \text{and} \qquad [\hat{B}_i^*, \hat{B}_j^*] = 0 \qquad i \neq j, 1, 2, 3. \tag{21}$$

The forms given by (19) and (20) furnish a particular solution to (11). In general the dispersion operators $\hat{B}_j^*$ determined from (11), may be operators and functions of the physical variables and the spectral parameter.

For simplicity, in the following sections, the dispersion operators $\hat{B}_j^*$ are always assumed to be multiplicative operators and the only case (a) in proposition 1 is considered below.

3. Equivalence between the different methods

In this section, we shall show that for a large class of dispersion relations, the $\bar{\partial}$-problem whose spectral transform operator R has an inhomogeneous time and space dependence, is, in fact, equivalent to a $\bar{\partial}$-problem with variable normalization. We consider the case where the operators $\hat{N}_i$, $i = 1, 2, 3$ take the form (12). It is clear that equation (6) degenerates to the usual form when the dispersion operators take a polynomial dependence on the spectral parameter. However, for general dispersion relations with singularities, it is not trivial.

At first, we consider the special case where the dispersion operators $\hat{B}_i^*$ assume the form

$$\hat{B}_i(\lambda) = I_i(\lambda) = \frac{A_i}{\lambda - \lambda_i} \qquad i = 1, 2, 3 \tag{22}$$

where the matrices A_i $(i = 1, 2, 3)$ commute

$$[A_i, A_j] = 0. \tag{23}$$

The dispersion relation selected, can be seen to satisfy (11). Accordingly, the kernels of the integral operators $\hat{N}_i$ (12) are now given by

$$N_i(\mu,\lambda) = \frac{-1}{2\mathrm{i}}\frac{\partial I_i(\lambda)}{\partial\bar{\lambda}}\delta(\mu-\lambda) \qquad i = 1, 2, 3. \tag{24}$$

By virtue of the identity

$$\frac{-1}{2\mathrm{i}}\iint_c \mathrm{d}\mu\wedge\mathrm{d}\bar{\mu}\,\chi(\mu,\bar{\mu})\,\delta(\lambda-\mu) = \chi(\lambda,\bar{\lambda}) \tag{25}$$

the action of integral operators $\hat{N}_i$ has the simple expression

$$\int \mathrm{d}\mu\wedge\mathrm{d}\bar{\mu}\,\chi(\mu,\bar{\mu})\,N_i(\mu,\bar{\mu};\lambda,\bar{\lambda}) = \frac{\partial B_i(\lambda)}{\partial\bar{\lambda}}\chi(\lambda,\bar{\lambda}). \tag{26}$$

It is easily proved that the operators defined in this way satisfy the constraint (10).

In this case, equation (7) can be rewritten as

$$\begin{aligned}\frac{\partial}{\partial x_i}R(\mu,\bar{\mu};\lambda,\bar{\lambda};x) &= I_i(\mu)\,R(\mu,\bar{\mu};\lambda,\bar{\lambda};x)\\ &\quad -R(\mu,\bar{\mu};\lambda,\bar{\lambda};x)\,I_i(\lambda) - \frac{1}{2\mathrm{i}}\frac{\partial I_i(\lambda)}{\partial\bar{\lambda}}\delta(\mu-\lambda).\end{aligned} \tag{27}$$

Solving this equation yields

$$\begin{aligned}R(\mu,\bar{\mu};\lambda,\bar{\lambda};x) &= \mathrm{e}^{F(\mu)}R_0(\mu,\bar{\mu};\lambda,\bar{\lambda})\,\mathrm{e}^{-F(\lambda)}\\ &\quad -\frac{1}{2\mathrm{i}}\mathrm{e}^{F(\mu)}\sum_i \partial_i^{-1}\left[\mathrm{e}^{-F(\mu)}\frac{\partial}{\partial\bar{\lambda}}I_i(\lambda)\,\mathrm{e}^{F(\lambda)}\right]\mathrm{e}^{-F(\lambda)}\end{aligned} \tag{28}$$

where the symbol ∂^{-1} denotes an integral operator, and where the matrix-valued function $F(\mu,x)$ is defined by

$$F(\mu,x) \equiv \sum_i \frac{A_i}{\mu-\lambda_i}x_i.$$

In the constructions we follow here we shall assume that the kernel R_0 is an integral operator kernel of Fredholm type and have no essential singularities at the points where I_i have poles.

By virtue of (23), it can be shown that

$$\partial_i^{-1}[\exp(-F(\mu)+F(\lambda))] = -\frac{(\mu-\lambda_i)(\lambda-\lambda_i)}{\lambda-\mu}A_i^{-1}\exp(-F(\mu)+F(\lambda)) + c_i \tag{29}$$

where c_i is a constant matrix which is a function of the spectral parameter only and which we take to be zero in the following sections.

Inserting (29) and (28) into (6) yields

$$\begin{aligned}\frac{\partial}{\partial\bar{\lambda}}\chi &= \frac{\partial\eta}{\partial\bar{\lambda}} + \iint_c \mathrm{d}\mu\wedge\mathrm{d}\bar{\lambda}\,\chi(\mu,\bar{\mu})\,R(\mu,\bar{\mu};\lambda,\bar{\lambda};x)\\ &= \frac{\partial\eta}{\partial\bar{\lambda}} + \iint_c \mathrm{d}\mu\wedge\mathrm{d}\bar{\mu}\,\chi(\mu,\bar{\mu})\,\mathrm{e}^{F(\mu)}\,R_0(\mu,\lambda)\,\mathrm{e}^{-F(\lambda)}\\ &\quad +\frac{1}{2\mathrm{i}}\sum_i\iint_c \mathrm{d}\mu\wedge\mathrm{d}\bar{\mu}\,\chi(\mu,\bar{\mu})\frac{(\mu-\lambda_i)(\lambda-\lambda_i)}{\lambda-\mu}A_i^{-1}\frac{\partial}{\partial\bar{\lambda}}B_i(\lambda)\delta(\mu-\lambda)\\ &\equiv \frac{\partial\eta}{\partial\bar{\lambda}} + I(1) + I(2).\end{aligned} \tag{30}$$

The crucial problem comes from $I(2)$ which has a very non-trivial role.

From the distribution identities [10]

$$\frac{\partial}{\partial\bar{\lambda}}\left(\frac{(-1)^n n!}{\pi}\frac{1}{(\lambda-\mu)^{n+1}}\right)=\delta^{(n,0)}(\lambda-\mu)\equiv\frac{\partial^n}{\partial\lambda^n}\delta(\lambda-\mu) \tag{31}$$

$$\frac{\partial}{\partial\bar{\lambda}}\left(\frac{1}{\lambda-\lambda_i}\right)=\pi\delta(\lambda-\lambda_i) \tag{32}$$

the second term in (30) can be rewritten as

$$\begin{aligned}I(2)&=-\frac{1}{4\mathrm{i}}\sum_{i=1}\iint_c \mathrm{d}\mu\wedge\mathrm{d}\bar{\mu}\,\chi(\mu,\bar{\mu})(\mu-\lambda_i)(\lambda-\lambda_i)\,\delta(\lambda-\lambda_i)\,\delta_\mu^{(1,0)}(\mu-\lambda)\\&=\frac{1}{2}\sum_{i=1}\left[\left(\frac{\partial}{\partial\lambda}\chi(\lambda,\bar{\lambda})\right)(\lambda-\lambda_i)^2\delta(\lambda-\lambda_i)+\chi(\lambda,\bar{\lambda})(\lambda-\lambda_i)\delta(\lambda-\lambda_i)\right].\end{aligned} \tag{33}$$

The role of the second term in (30) depends then on the behaviour of function χ in the vicinity of λ_i ($i=1,2,3$). It can be seen that the different behaviour of function χ in the vicinity of λ_I ($i=1,2,3$), will induce different normalizations at these special points. This situation is different from the usual $\bar{\partial}$-problem with variable normalization because the inhomogeneous term cannot be arbitrarily selected, but must be determined by the term $I(2)$.

Case A. χ regular at λ_I ($i=1,2,3$). If the function χ is regular at λ_I ($i=1,2,3$), we then have

$$\frac{\partial\eta}{\partial\bar{\lambda}}=0\qquad\chi(\lambda)\to a_i+\mathrm{O}(\lambda-\lambda_i). \tag{34}$$

In this case, expression (33) vanishes automatically. To make the problem (6) uniquely solvable, we can take the canonical normalization

$$\chi(\lambda)\to 1\qquad\text{as}\quad\lambda\to\infty. \tag{35}$$

Case B: χ singular at λ_I ($i=1,2,3$). Supposing that the function χ has poles at λ_I ($i=1,2,3$), namely

$$\chi_k(\lambda)=\frac{a_k}{(\lambda-\lambda_k)^m}+\mathrm{O}(\lambda-\lambda_k)\qquad m\geqslant 1\quad k=1,2,3. \tag{36}$$

Expression (33) becomes

$$\begin{aligned}&\frac{-1}{4\mathrm{i}}\sum_{i=1}\iint_c \mathrm{d}\mu\wedge\mathrm{d}\bar{\mu}\,\chi(\mu,\bar{\mu})(\mu-\lambda_i)(\lambda-\lambda_i)\,\delta(\lambda-\lambda_i)\,\delta^{(1,0)}(\mu-\lambda)\\&\qquad=\frac{1}{2}\sum_{i=1}\left[\frac{-m}{(\lambda-\lambda_k)^{m+1}}(\lambda-\lambda_i)^2+\frac{1}{(\lambda-\lambda_k)^m}(\lambda-\lambda_i)\right]a_k\delta(\lambda-\lambda_i)\\&\qquad\lambda\to\lambda_k.\end{aligned} \tag{37}$$

Inserting (37) into (30) yields

$$\begin{aligned}\frac{\partial}{\partial\bar{\lambda}}\chi&=\frac{\partial\eta}{\partial\bar{\lambda}}+\iint_c \mathrm{d}\mu\wedge\mathrm{d}\bar{\lambda}\,\chi(\mu,\bar{\mu})\,R(\mu,\bar{\mu};\lambda,\bar{\lambda};x)\\&=\iint_c \mathrm{d}\mu\wedge\mathrm{d}\bar{\mu}\,\chi(\mu,\bar{\mu})\,\mathrm{e}^{F(\mu)}R_0(\mu,\lambda)\,\mathrm{e}^{-F(\lambda)}+\frac{\partial\eta}{\partial\bar{\lambda}}\\&\quad+\frac{1}{2}\sum_i\left[\frac{(-m)}{(\lambda-\lambda_k)^{m+1}}(\lambda-\lambda_i)^2+\frac{1}{(\lambda-\lambda_k)^m}(\lambda-\lambda_i)^1\right]a_i\delta(\lambda-\lambda_i)\\&\quad\lambda\to\lambda_k.\end{aligned} \tag{38}$$

For $m = 1$, the last term in (38) vanishes automatically. However, when $m \neq 1$, this term does not vanish and in the vicinity of $\lambda \to \lambda_I$, the rational function η is determined by requiring the following relation:

$$\frac{\partial}{\partial\bar\lambda}\chi \to \frac{\partial\eta}{\partial\bar\lambda} + \frac{1}{2}\frac{(1-m)}{(\lambda-\lambda_i)^{m-1}}a_i\delta(\lambda-\lambda_i) \qquad \lambda \to \lambda_i. \tag{39a}$$

This means that

$$\frac{\partial\eta}{\partial\bar\lambda} \to \frac{3}{2}\frac{(m-1)}{(\lambda-\lambda_i)}a_i\delta(\lambda-\lambda_i) \qquad \lambda \to \lambda_i. \tag{39b}$$

Consequently, the $\bar\partial$-problem (30) is transformed into the equivalent form

$$\frac{\partial}{\partial\bar\lambda}\chi(\lambda,\bar\lambda) = \frac{\partial\eta'}{\partial\bar\lambda} + \hat R'\chi = \frac{\partial\eta'}{\partial\bar\lambda} + \iint_c \mathrm{d}\lambda\wedge\mathrm{d}\bar\lambda\,\chi(\mu,\bar\mu)R'(\mu,\bar\mu;\lambda,\bar\lambda;x) \tag{40}$$

where the integral operator R' satisfies the following dependence on the physical variables x_i:

$$\frac{\partial}{\partial x_i}\hat R' = [\hat R', \hat B_i^*] \qquad i = 1, 2, 3 \tag{41}$$

and

$$\frac{\partial}{\partial\bar\lambda}\chi \to \frac{\partial\eta'}{\partial\bar\lambda}. \tag{42}$$

Without loss of generality, the general dispersion relation with poles can be taken as

$$\hat B_i(\lambda) = I_i(\lambda) = \sum_\alpha \frac{A_{\alpha(i)}}{\lambda-\lambda_{\alpha(i)}} \tag{43}$$

where the matrices $A_{a(i)}$ ($i = 1, 2, 3$) commute. In the same way, we can show that the $\bar\partial$-problem with dispersion relations (43) can be transformed into the corresponding one with the physical variable dependence of R' of form (41). From the above analysis, we have the following theorem.

Theorem 2. *For general multiplicative dispersion relation with the form (43), Leon's extension for the inverse spectral transform is equivalent to the $\bar\partial$-problem with a variable normalization.*

Before closing this section, we want to emphasize that a number of nonlinear partial differential equations can be constructed by the $\bar\partial$-dressing method whose dispersion relation takes the form (43). For example, the Zakharov–Manakov system has the dispersion operators

$$\hat B_i(\lambda) = I_i(\lambda) = \frac{A_i}{\lambda-\lambda_i} \qquad i = 1, 2, 3 \tag{44}$$

the KP equation has

$$\hat B_1(\lambda) = \frac{1}{\lambda} \qquad \hat B_2(\lambda) = \frac{1}{2\varepsilon}[(\lambda-\varepsilon)^{-1} - (\lambda+\varepsilon)^{-1}]$$
$$\hat B_3(\lambda) = \frac{1}{2\varepsilon^2}[(\lambda-\varepsilon)^{-1} + (\lambda+\varepsilon)^{-1}] - 2\lambda^{-1}]. \tag{45}$$

and the $(2 + 1)$-dimensional sine–Gordon equation has

$$\hat B_1(\lambda) = \mathrm{i}\lambda \qquad \hat B_2(\lambda) = -\mathrm{i}\lambda\frac{1}{\sigma}\sigma_3 \qquad \hat B_3(\lambda) = -\frac{\mathrm{i}\sigma}{2\lambda}m. \tag{46}$$

In the above formula, σ_3 is the Pauli matrix, ε is a scalar factor and m a 2×2 diagonal matrix. In the monograph [9], several well known nonlinear evolution systems have been constructed.

4. Concluding remarks

In conclusion, we have extended Leon's idea to the (2 + 1)-dimensional problem and shown that for general x-independent dispersion relations with poles, Leon's extension for the inverse spectral transform is equivalent to the $\bar{\partial}$-problem with variable normalization, at least in a subclass of dispersion relations.

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